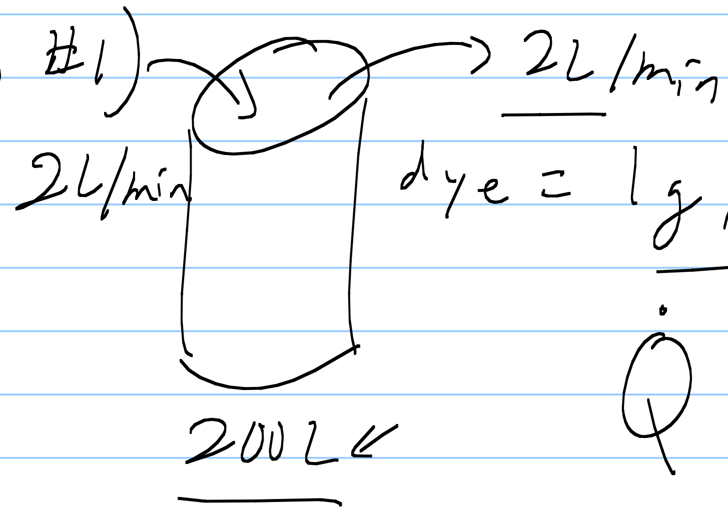
MAT 227 6/4

Note Title 6/4/2009

2.3 #1)



dye = 1g/L $Q(0) = 200 \cdot 1 = 200g$

$\dot{Q} = IN - OUT$

$= 0 - \frac{Q}{V} \cdot r$

$\frac{dQ}{dt} = -\frac{1}{100} Q$ $\frac{dQ}{Q} = -\frac{1}{100} dt \rightarrow Q(t) = Ae^{-t/100}$

$Q(t) = 200e^{-t/100} = 2$

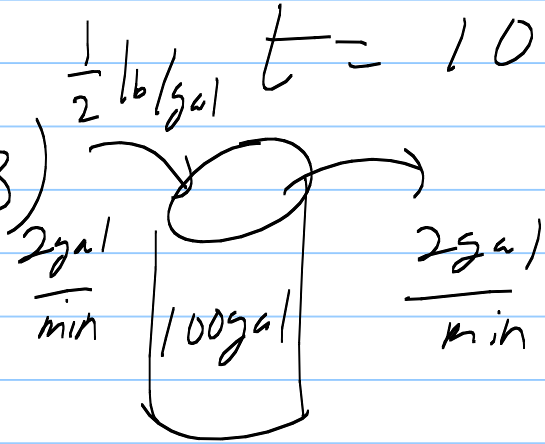
$$\ln(Q) = -\frac{1}{100}t + C \qquad Q = Ae^{-t/100}$$

$$e^{-t/100} = .01$$

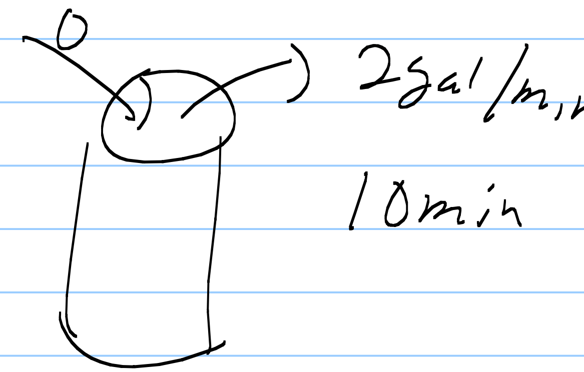
$$-\frac{t}{100} = \ln(.01) = -\ln(100)$$

$$t = 100\ln(100) = 460.5 \text{ min}$$

2.3 #3)



for 10min



$$\dot{Q} = IN - OUT = \left(\frac{1}{2}\right)(2) - \frac{Q}{100} \cdot 2$$

$$\dot{Q} = 1 - \frac{1}{50} Q \qquad Q(0) = 0$$

$$= -\frac{1}{50}\left[Q - 50\right]$$

$$\frac{dQ}{Q - 50} = -\frac{1}{50} dt$$

$$\ln(Q - 50) = -\frac{t}{50} + c \rightarrow Q = 50 + Ae^{-t/50}$$

$$Q(0) = 0 \Rightarrow 50 + A = 0 \Rightarrow A = -50$$

$$Q(t) = 50\left[1 - e^{-t/50}\right] \qquad 0 < t < 10$$

$$Q(10) = 50\left[1 - e^{-.2}\right] \doteq 9.1$$

$$\dot{Q} = IN - OUT = 0 - \frac{Q}{100} \cdot 2$$

$$\dot{Q} = -\frac{1}{50} Q \implies Q(t) = 9.1 e^{-\frac{t-10}{50}}$$

$$\underline{Q(10) = 9.1}$$

$$Q(20) = 9.1 e^{-\frac{1}{5}} = 7.4 \text{ lbs}$$

$$y' = ay$$

$$y(t_0) = y_0$$

$$\frac{dy}{y} = a\,dt$$

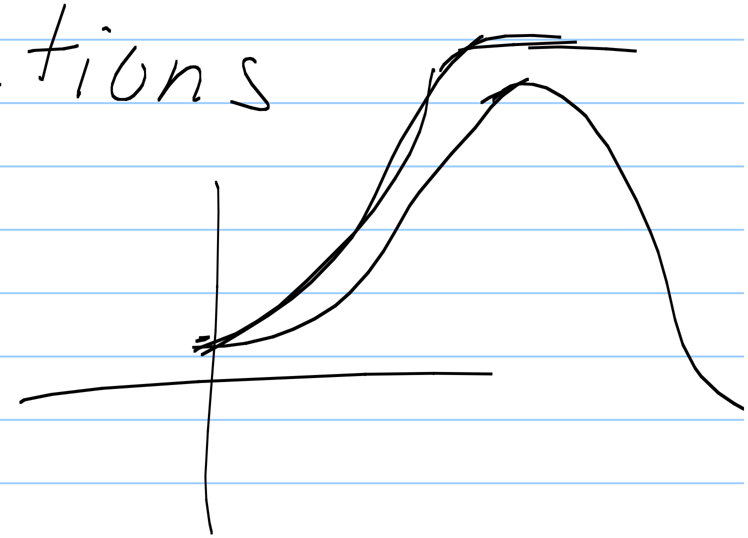
$$\ln(y) = at + c$$

$$y = Ae^{at}$$

$$y_0 = Ae^{at_0} \quad A = y_0 e^{-at_0}$$

$$y(t) = y_0 e^{a(t-t_0)}$$

Autonomous Equations

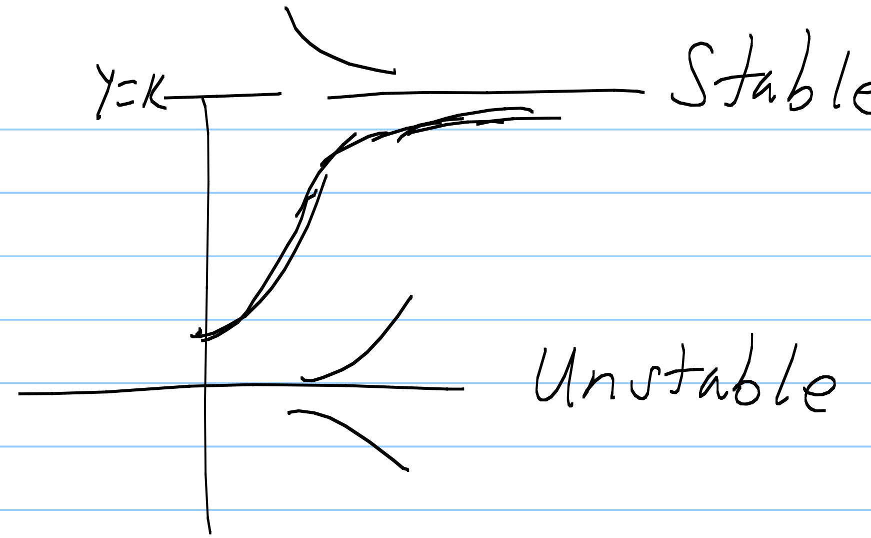


$$y' = f(y)$$

$$y' = ry, \quad y(0) = y_0 \implies y(t) = y_0 e^{rt}$$

$$y' = ry\left(1 - \frac{y}{K}\right)$$

$\to 0$



$$ry\left(1 - \frac{y}{K}\right) = 0$$

$\hookrightarrow y = K$

$\hookrightarrow y = 0$

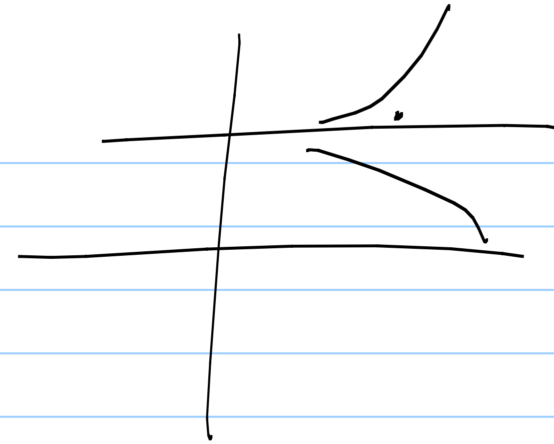
$f(y) = 0$

Critical points

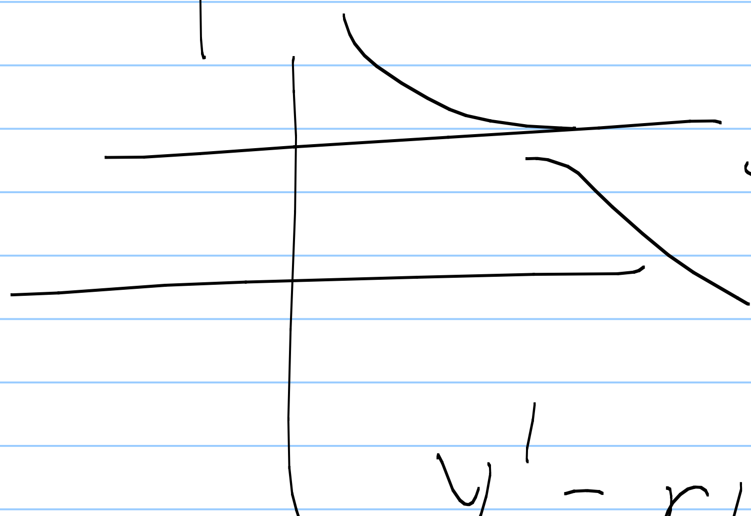
Sol'n to ODE

Stable or Unstable or Semistable

stable



unstable



Semistable

$$y' = ry\left(1-\frac{y}{K}\right)$$

$$\frac{dy}{y\left(1-\frac{y}{K}\right)} = r\,dt$$

$$\frac{1}{y(1-\frac{y}{K})} = \frac{1}{y} + \frac{1}{K(1-y/K)}$$

$$\frac{dy}{y} + \frac{dy}{K(1-y/K)} = r\,dt$$

$$\ln(y) - \ln(K-y) = rt + C$$

$$\ln\left[\frac{y}{K-y}\right] = rt + C$$

$$\left(\frac{y}{K-y} = Ae^{rt}\right)(K-y)$$

$t=0,\ y=y_0 \Rightarrow A = \frac{y_0}{K-y_0}$

$$y = AKe^{rt} - Aye^{rt}$$

$$y\left[1+Ae^{rt}\right] = AKe^{rt}$$

$$y = \frac{AKe^{rt}}{1+Ae^{rt}}$$

$$y = \frac{\frac{y_0}{\cancel{K-y_0}} K \cancel{e^{rt}}}{1 + \frac{y_0}{\cancel{K-y_0}} e^{rt}} \cdot \frac{\cancel{(K-y_0)}\cancel{e^{-rt}}}{(K-y_0)e^{-rt}}$$

$$\boxed{y(t) = \frac{K y_0}{y_0 + (K-y_0)e^{-rt}}} \Leftarrow$$

$y_0 = K \Rightarrow y(t) = K$

$y_0 = 0 \Rightarrow y(t) = 0$

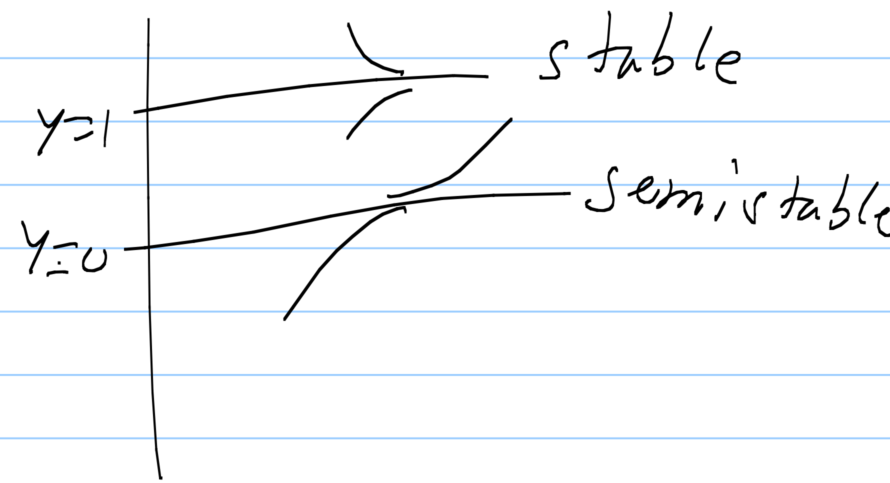
$$y' = y^2(1-y) = 0 \Rightarrow y = 0, \quad y = 1$$

$y < 0 \quad y^2(1-y) > 0$

$y > 0 \quad y^2(1-y) > 0$

$y < 1 \quad y^2(1-y) > 0$

$y > 1 \quad y^2(1-y) < 0$



$$\int \frac{dy}{y^2(1-y)} = \int dt \qquad \frac{1}{y^2(1-y)} = \frac{1}{y^2} + \frac{1}{1-y} + \frac{1}{y}$$

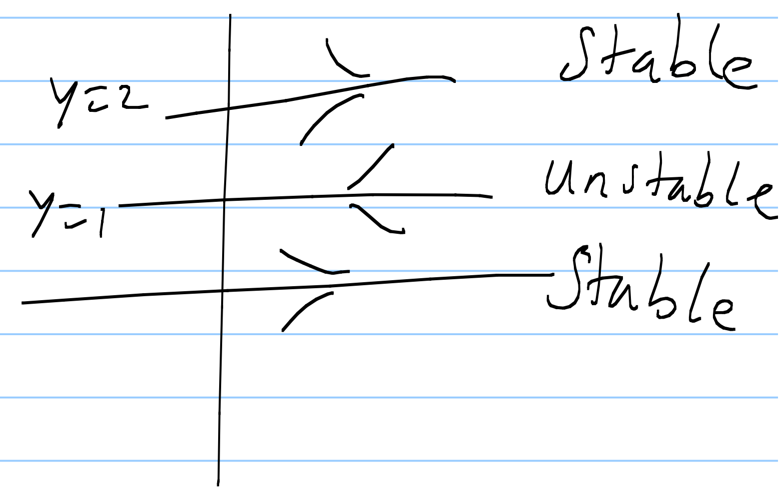
$$\frac{dy}{y^2} + \frac{dy}{1-y} + \frac{dy}{y} = dt$$

$$-\frac{1}{y} - \ln(1-y) + \ln(y) = t + c$$

$$\frac{y}{1-y} e^{-1/y} = Ae^{t}$$

$$y' = -y(1-y)(2-y) = 0$$

$y = 0, 1, 2$



$$\frac{dy}{y(1-y)(2-y)} = -dt$$

$$\frac{1}{y(1-y)(2-y)} = \frac{A}{y} + \frac{B}{1-y} + \frac{C}{2-y}$$

$$= \frac{1}{2y} + \frac{1}{1-y} - \frac{1}{2(2-y)}$$

$$A = \frac{1}{(1-y)(2-y)}\Big|_{y=0} = \frac{1}{2}$$

$$B = \frac{1}{y(2-y)}\Big|_{y=1} = 1$$

$$C = \frac{1}{y(1-y)}\Big|_{y=2} = -\frac{1}{2}$$

$$\left[\frac{1}{y(1-y)(2-y)} = \frac{A}{y} + \frac{B}{1-y} + \frac{C}{2-y}\right](2-y)$$

$$\left.\frac{1}{y(1-y)}\right|_{y=2} = A\frac{2-y}{y} + B\frac{2-y}{1-y} + C \quad @ \; \underline{y=2}$$

$$\frac{dy}{2y} + \frac{dy}{1-y} - \frac{dy}{2(2-y)} = -dt$$

$$\frac{1}{2}\ln(y) - \ln(1-y) + \frac{1}{2}\ln(2-y) = -t + C$$

$$\ln\left[\frac{\sqrt{y(2-y)}}{1-y}\right] = -t + C$$

$$\frac{\sqrt{y(2-y)}}{1-y} = Ae^{-t}$$

$$\frac{y(2-y)}{(1-y)^2} = Ae^{-2t}$$

$$2y - y^2 = Ae^{-2t}(1 - 2y + y^2)$$

$$y^2\left[1 + Ae^{-2t}\right] - 2y\left[1 + Ae^{-2t}\right] + Ae^{-2t} = 0$$

$$y(t) = \frac{\left[1 + Ae^{-2t}\right] \pm \left[(1 + Ae^{-2t})^2 - Ae^{-2t}(1 + Ae^{-2t})\right]^{1/2}}{1 + Ae^{-2t}}$$

$$1 + \cancel{2}Ae^{-2t} + \cancel{A^2 e^{-4t}} - \cancel{Ae^{-2t}} - \cancel{A^2 e^{-4t}}$$

$$y(t) = \frac{1 + Ae^{-2t} \pm \sqrt{1 + Ae^{-2t}}}{1 + Ae^{-2t}}$$

$$y(t) = 1 \pm \left[1 + Ae^{-2t}\right]^{-1/2}$$