Chapter 5 – Common Substitution Summary Table

	Forms Involving	Substitution	dx	Replace	
1	$\sqrt{a^2 - x^2}$	$x = a\sin(t)$	$dx = a\cos(t)\,dt$	$\sqrt{a^2 - x^2} = a\cos(t)$	$t = \sin^{-1}\left(x/a\right)$
2	$\sqrt{a^2 + x^2}$	$x = a\sinh(t)$	$dx = a\cosh(t)\,dt$	$\sqrt{a^2 + x^2} = a\cosh(t)$	$t = \sinh^{-1}\left(x/a\right) = \ln\left[\dfrac{x + \sqrt{a^2 + x^2}}{a}\right]$
3	$\sqrt{x^2 - a^2}$	$x = a\cosh(t)$	$dx = a\sinh(t)\,dt$	$\sqrt{x^2 - a^2} = a\sinh(t)$	$t = \cosh^{-1}\left(x/a\right) = \ln\left[\dfrac{x + \sqrt{x^2 - a^2}}{a}\right]$
4	$\ln(x)$	$u = \ln(x)$	$du = \dfrac{dx}{x}$	$x = e^u$	
5	e^{ax}	$u = e^{ax}$	$du = e^{ax}dx$	$x = \dfrac{\ln(u)}{a}$	

Useful Identities

$1-\cos^2(t)=\sin^2(t)$	$1-\sin^2(t)=\cos^2(t)$	$\cos^2(t) = \dfrac{1+\cos(2t)}{2}$	$\sin^2(t) = \dfrac{1-\cos(2t)}{2}$	$\sin(2t) = 2\sin(t)\cos(t)$
$\cosh^2(t)-1=\sinh^2(t)$	$1+\sinh^2(t)=\cosh^2(t)$	$\cosh^2(t) = \dfrac{1+\cosh(2t)}{2}$	$\sinh^2(t) = \dfrac{\cosh(2t)-1}{2}$	$\sinh(2t) = 2\sinh(t)\cosh(t)$

Examples

Using (1)	$\int\sqrt{a^2-x^2}\,dx = a^2\int\cos^2(t)dt = \dfrac{a^2}{2}\int[1+\cos(2t)]dt = \dfrac{a^2}{2}t + \dfrac{a^2}{4}\sin(2t) + c = \dfrac{a^2}{2}t + \dfrac{a^2}{2}\sin(t)\cos(t) + c = \dfrac{a^2}{2}\sin^{-1}(x/a) + \dfrac{x}{2}\sqrt{a^2-x^2} + c$
Using (2)	$\int\sqrt{a^2+x^2}\,dx = a^2\int\cosh^2(t)dt = \dfrac{a^2}{2}\int[1+\cosh(2t)]dt = \dfrac{a^2}{2}t + \dfrac{a^2}{4}\sinh(2t) + c = \dfrac{a^2t}{2} + \dfrac{a^2}{2}\sinh(t)\cosh(t) + c = \dfrac{a^2}{2}\sinh^{-1}(x/a) + \dfrac{x}{2}\sqrt{a^2+x^2} + c$
Using (3)	$\int\sqrt{x^2-a^2}\,dx = a^2\int\sinh^2(t)dt = \dfrac{a^2}{2}\int[\cosh(2t)-1]dt = \dfrac{a^2}{4}\sinh(2t) - \dfrac{a^2}{2}t + c = \dfrac{a^2}{2}\sinh(t)\cosh(t) - \dfrac{a^2t}{2} + c = \dfrac{x}{2}\sqrt{x^2-a^2} - \dfrac{a^2}{2}\cosh^{-1}(x/a) + c$
Using (1)	$\int\dfrac{x^2}{\sqrt{a^2-x^2}}dx = a^2\int\sin^2(t)dt = \dfrac{a^2}{2}\int[1-\cos(2t)]dt = \dfrac{a^2}{2}t - \dfrac{a^2}{4}\sin(2t) + c = \dfrac{a^2}{2}t - \dfrac{a^2}{2}\sin(t)\cos(t) + c = \dfrac{a^2}{2}\sin^{-1}(x/a) - \dfrac{x}{2}\sqrt{a^2-x^2} + c$
Using (2)	$\int\dfrac{x^2}{\sqrt{a^2+x^2}}dx = a^2\int\sinh^2(t)dt = \dfrac{a^2}{2}\int[\cosh(2t)-1]dt = \dfrac{a^2}{4}\sinh(2t) - \dfrac{a^2}{2}t + c = \dfrac{a^2}{2}\sinh(t)\cosh(t) - \dfrac{a^2}{2}t + c = \dfrac{x}{2}\sqrt{a^2+x^2} - \dfrac{a^2}{2}\sinh^{-1}(x/a) + c$
Using (3)	$\int\dfrac{x^2}{\sqrt{x^2-a^2}}dx = a^2\int\cosh^2(t)dt = \dfrac{a^2}{2}\int[\cosh(2t)+1]dt = \dfrac{a^2}{4}\sinh(2t) + \dfrac{a^2}{2}t + c = \dfrac{a^2}{2}\sinh(t)\cosh(t) + \dfrac{a^2t}{2} + c = \dfrac{x}{2}\sqrt{x^2-a^2} + \dfrac{a^2}{2}\cosh^{-1}(x/a) + c$